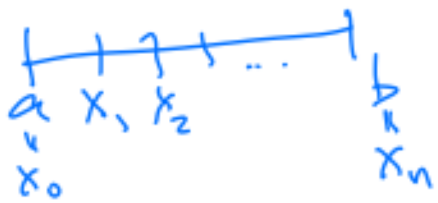
Quadrature, continued

Fancy name for numerical integration.

General Question: $\int_a^b f(x)\,dx$ ← how do we approximate this?

Next attempt: substitute degree 1 Newton finite difference formula.



divide interval into n evenly spaced pieces

$$h = \Delta x = \frac{b-a}{n}$$

$$x_j = a + j\,\Delta x = a + jh$$

$$y_j = f(x_j).$$

Part of our integral:

$$\int_{x_0}^{x_1} f(x)\,dx \overset{\text{F.D. formula}}{=} \int_{x_0}^{x_1} y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{f''(c)}{2!}(x-x_0)(x-x_1)\,dx$$

for some c between x_0 & x_1

$$= \int_{x_0}^{x_1} y_0\,dx + \int_{x_0}^{x_1} \frac{\Delta y_0}{h}(x-x_0)\,dx + \int_{x_0}^{x_1} \frac{f''(c)}{2}(x-x_0)(x-x_1)\,dx$$

u substitution: Let $u = \frac{x-x_0}{h}$

bounds: $x = x_0 \rightarrow u = 0$

$x = x_1 \rightarrow u = \frac{x_1 - x_0}{h} = \frac{h}{h} = 1$

$$du = \frac{dx}{h} \Rightarrow dx = h\,du$$

$$= y_0(x_1 - x_0) + \int_{u=0}^{1} (\Delta y_0)\, u \cdot h\, du + \int_{u=0}^{1} \frac{f''(c)}{2}(hu)\,h(u-1)\,h\,du$$

$$u = \frac{x - x_0}{h} \Rightarrow x - x_0 = hu$$

$$x - x_1 = x - x_0 - (x_1 - x_0)$$

$$\Rightarrow x - x_1 = hu - h = h(u-1)$$

$$= y_0 h + (\Delta y_0) h \underbrace{\left(\int_0^1 u\,du\right)}_{\frac{u^2}{2}\Big|_0^1 = \frac{1}{2}} + \frac{h^3}{2} \int_{u=0}^{1} f''(c)\, u(u-1)\,du$$

$u(u-1)$ is negative for $u \in (0,1)$

$$M u(u-1) \le f''(c) \underbrace{u(u-1)}_{\text{negative}} \le m\, u(u-1)$$

max value.

minimum value of $f''(c)$ for $x_0 \le c \le x_1$

integrate all

$$M \int_0^1 u(u-1)\,du \le \int_0^1 f''(c)\, u(u-1)\,du \le m \int u(u-1)\,du$$

$$\int u^2 - u$$

$$\frac{u^3}{3} - \frac{u^2}{2}\Big|_0^1$$

$$\frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

$$-\frac{1}{6}M \le \int_0^1 f''(c)\, u(u-1)\,du \le -\frac{1}{6}m$$

by intermediate value thm, if f'' is continuous,

$$\int_0^1 f''(c)\, u(u-1)\,du = -\frac{1}{6} f''(\xi_1)$$

$$\Rightarrow \int_{x_0}^{x_1} f(x)\,dx = y_0 h + \Delta y_0 h\left(\frac{1}{2}\right) + \frac{h^3}{2}\left(-\frac{1}{6} f''(\xi_1)\right)$$

$$= y_0 h + \frac{(y_1 - y_0)h}{2} + -\frac{h^3}{12} f''(\xi_1)$$

$$\Rightarrow \int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{y_1}{2} - \frac{y_0}{2}\right) + -\frac{h^3}{12} f''(\xi_1)$$

$$\Rightarrow \int_{x_0}^{x_1} f(x)\,dx = \underbrace{h\left(\frac{y_0}{2} + \frac{y_1}{2}\right)}_{\substack{\text{Trap rule} \\ \text{for } [x_0, x_1]}} - \frac{h^3}{12} f''(\xi).$$

Now we add these formulas

$$\int_{x_k}^{x_{k+1}} f(x)\,dx = h\left(\frac{y_k}{2} + \frac{y_{k+1}}{2}\right) - \frac{h^3}{12} f''(\xi_{k+1})$$

Add from $k = 0$ to $k = n-1$

$$\Rightarrow \int_a^b f(x)\,dx = \underbrace{\sum_{k=0}^{n-1} \frac{h}{2}(y_k + y_{k+1})}_{\text{TRAP}(n).} - \underbrace{\frac{h^3}{12} \sum_{k=0}^{n-1} f''(\xi_{k+1})}_{\substack{\text{error for} \\ \text{Trap}(n).}}$$

Similar trick as before:

$$\sum_{k=0}^{n-1} f''(\xi_{k+1}) = n \cdot \left(\underbrace{\frac{1}{n}\sum_{k=0}^{n-1} f''(\xi_{k+1})}_{\text{avg of } f'' \text{ at } n \text{ pts in } [a,b]} \right)$$

By intermediate value thm, it is $= f''(\tau)$ for some $\tau \in [a,b]$.

$$\Rightarrow \int_a^b f(x)\,dx = \underbrace{\text{Trap}(n)}_{\frac{h}{2}(y_0+2y_1+2y_2+\dots+2y_{n-1}+y_n)} + \underbrace{-\frac{h^3}{12} n f''(\tau)}_{\text{Error in Trap}(n)}$$

$$h = \frac{b-a}{n}$$

$$\Rightarrow \text{Error in Trap}(n) = -\frac{\left(\frac{b-a}{n}\right)^3 \cdot n}{12} f''(\tau)$$

$$= \boxed{-\frac{(b-a)^3}{12n^2} f''(\tau)}$$

for some $\tau \in [a,b]$.

n^2 in denom $\Rightarrow$ Trap(n) converges faster to $\int_a^b f(x)\,dx$ as $n \to \infty$

than Left(n) or Right(n).

Upshot:

$$\int f(x)\,dx = \underbrace{\int \left(\text{FD formula of degree } m\right)}_{\text{gives integral approximation}} + \underbrace{\left(\text{FD error formula for degree } m\right)}_{\text{gives error formula}} dx$$

If all we want is a formula to estimate the integral, we can look at just the first part!

Let's find the next level approximation:

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0} \left(\text{FD approx of degree 2}\right) dx$$

$$\int_{x_0}^{x_2} y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2h^2}(x-x_0)(x-x_1)\,dx$$

Substitute $u = \dfrac{x-x_0}{h}$

$$du = \frac{dx}{h} \Rightarrow dx = h\,du$$

$(x-x_0) = hu$

$(x-x_1) = x-x_0-(x_1-x_0) = hu-h = h(u-1)$

Bounds $\begin{cases} x = x_0 & u = \frac{x - x_0}{h} = 0 \\ x = x_2 & u = \frac{x_2 - x_0}{h} = \frac{2h}{h} = 2 \end{cases}$

$$\Rightarrow \int_{x_0}^{x_2} f(x)\,dx \approx y_0(x_2 - x_0) + \int_{u=0}^{2} \left[\Delta y_0 u + \frac{\Delta^2 y_0}{2h^2}(hu)(h(u-1)) \right] h\,du$$

$$= y_0(x_2 - x_0) + (y_1 - y_0)h \underbrace{\int_0^2 u\,du}_{\frac{u^2}{2}\Big|_0^2 = 2} + \frac{\Delta^2 y_0}{2} h \int_0^2 u(u-1)\,du$$

$$\int u^2 - u$$

$$\frac{u^3}{3} - \frac{u^2}{2}\Big|_0^2$$

$$\underbrace{\frac{8}{3} - 2}_{\frac{2}{3}}$$

$$\int_{x_0}^{x_2} f(x)\,dx = y_0(2h) + (y_1 - y_0)(2h)$$

$$+ \frac{h}{\cancel{2}}(y_2 - 2y_1 + y_0)\frac{\cancel{2}}{3}$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$= y_2 - y_1 - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0$$

$$= h\left[\cancel{2y_0} + 2y_1 - \cancel{2y_0} + \frac{1}{3}y_2 - \frac{2}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= h\left[\underbrace{\left(2 - \frac{2}{3}\right)}_{\frac{4}{3}} y_1 + \frac{1}{3}y_2 + \frac{y_0}{3}\right] = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$\Rightarrow \int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} + \int_{x_2}^{x_4} + \dots \int_{x_{n-2}}^{x_n}$$

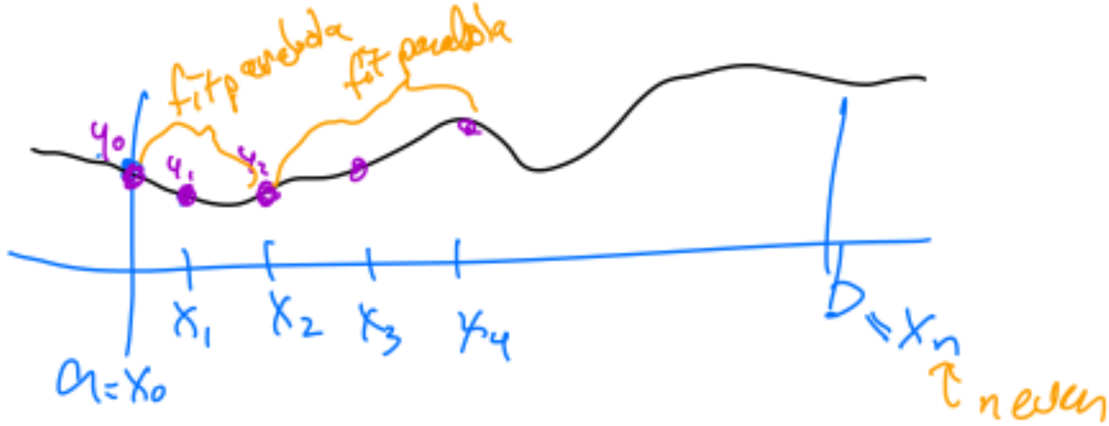
n must be even

$$\approx \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

$$+ \frac{h}{3}\left(y_4 + 4y_5 + y_6\right) \dots \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

"Simpson's Rule"



Error formula for Simp(n) (from text)

$$= -\frac{(b-a)^5}{180n^4} f''''(\tau)$$

n^4 makes this converge much faster than Trap(n) as $n \to \infty$.